Special and General Relativity corrections to the OPERA neutrino velocity measurement.

Elias Kiritsis[a,b] **and Fransesco Nitti**[b]

[a] *Crete Center for Theoretical Physics, Department of Physics, University of Crete, 71003 Heraklion, Greece*

[b] *APC, Université Paris 7, Bâtiment Condorcet, F-75205, Paris Cedex 13, France (UMR du CNRS 7164).*

Abstract: We calculate, corrections to the OPERA result due to the rotation of the earth (Sagnac effect), the Mass of the earth, moon, sun and the galaxy, and frame dragging effects. All of these are found to be negligible, and do not affect the experimental error quoted in the experiment.

Keywords: .

Contents

1. Introduction

We will discuss here several relativistic effects that affect the distance, and time of arrival of neutrinos from CERN to GranSasso relevant for the OPERA experiment. OPERA finds that neutrinos arrive 60 ± 12 ns earlier than they should, [1].

They include

1. The Sagnac effect, due to the rotation of the earth

2. The gravitational effect of the mass of the earth, the moon, the sun and the galaxy. The geodesic must be evaluated in the appropriate Schwartzschild metric.

3. A combination of the two (1+2) by using the Kerr metric, that includes the extra frame-dragging effect

   Appart from the Sagnac effect whose size is small but of the same order of magnitude as the experimental signal, all of the rest are found to be negligible.

2. The Sagnac effect

2.1 The disk case

A simplified two dimensional version of the effect that lead to its first experimental discovery is as follows: Lightrays are moving around in a disc at a path at fixed radius R, in two opposite directions. The disk rotates with angular velocity ω. There is a difference in the times needed for a circumnavigation of the circle in the two opposite directions equal to

$$\Delta t = \frac{4\pi R^2 \omega}{c^2 - \omega^2 R^2} \simeq \frac{4\pi R^2 \omega}{c^2} \tag{2.1}$$ 1

To obtain this we change from the standard flat Minkowski metric on the plane

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 \tag{2.2}$$ 2

to the coordinate system of the rotating disk

$$x = r\cos(\omega t + \theta) \quad , \quad x = r\sin(\omega t + \theta) \tag{2.3}$$ 3

so that the metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + r^2(\omega dt + d\theta)^2 \tag{2.4}$$ 4

The radial geodesics for light have $dr = 0$ and $ds = 0$. We obtain the comoving geodesics

$$cdt = R(\pm\omega dt + d\theta) \quad , \quad \frac{d\theta}{dt} = \frac{c \pm \omega R}{R} \tag{2.5}$$ 5

which we can easily integrate. Therefore the path along(opposite) the rotation is given by

$$\theta_\pm = \frac{t(c \pm \omega R)}{R} + \theta_0 \tag{2.6}$$ 6

and the time to go around once is

$$t_\pm = \frac{2\pi R}{c \mp \omega R} \tag{2.7}$$ 7

so that the time difference is

$$\Delta t = t_- - t_+ = \frac{4\pi R^2 \omega}{c^2 - \omega^2 R^2} \simeq \frac{4\pi R^2 \omega}{c^2} \tag{2.8}$$ 8

where the approximation assumes $\frac{\omega R}{c} \ll 1$.

If instead of a whole turn, we consider light, moving a distance $L << R$ in one direction, then

$$|\Delta t| \simeq \frac{\omega L R}{c^2} \tag{2.9}$$ 9

2.2 The spherical case

We now use the spherical metric relevant for the case of the rotating earth.

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.10}$$ 10

Consider the motion of the earth along the ϕ coordinate with angular velocity ω. In the earth-fixed system the metric is

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta(\omega dt + d\phi)^2) \tag{2.11}$$ 11

In cartesian coordinates for a rotating system around the z-axis we obtain, equivalently

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 + 2\omega(xdy - ydx)dt + \omega^2(x^2+y^2)dt^2 \tag{2.12}$$ 12

Consider two points on the sphere,

$$\vec{r_i} \equiv \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = r_i \begin{pmatrix} \sin\theta_i\cos\phi_i \\ \sin\theta_i\sin\phi_i \\ \cos\theta_i \end{pmatrix} \tag{2.13}$$ 13

The distance at straight line is $d_s = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$. The distance traveled in the inertial coordinate system is

$$d_i = |\hat{\vec{r}}_1(t+\Delta t) - \hat{\vec{r}}_2(t)| \tag{2.14}$$ 14

where the hat stands for the coordinates in the non-rotating system. We neglect here the relativistic correction that is of relative order $\frac{\omega_e R_e}{c} \simeq 1.5 \times 10^{-6}$.

The difference between $\hat{\vec{r}}_1$ and $\hat{\vec{r}}_2$ is a rotation by an angle $\omega\Delta t$. This angle for the CERN-GS experiment is

$$\omega_e \Delta t_0 \simeq 1.8 \cdot 10^{-7} \ll 1 \tag{2.15}$$

Selecting appropriately the coordinate system we have $\hat{\vec{r}}_2 = \vec{r}_2$ and

$$\begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \\ \hat{z}_1 \end{pmatrix} \simeq \begin{pmatrix} 1 & \omega\Delta t & 0 \\ -\omega\Delta t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_1 - \omega\Delta t\ y_1 \\ y_1 + \omega\Delta t\ x_1 \\ z_1 \end{pmatrix} \tag{2.16}$$ 15

Therefore

$$d_i \simeq d_s + \frac{\omega\Delta t(y_1x_2 - x_1y_2)}{d_s} + \mathcal{O}(\Delta t^2) = d_s + \frac{\omega\Delta t\ r_1\ r_2\ \sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)}{d_s} + \mathcal{O}(\Delta t^2) \tag{2.17}$$ 16

so that

$$d_i - d_s \simeq \frac{\omega\Delta t\ r_1\ r_2\ \sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2)}{d_s} = \vec{\omega}\cdot(\vec{r}_1 \times \vec{r}_2)\frac{\Delta t}{d_s} \tag{2.18}$$ 17

We use $\Delta t \simeq \frac{d_s}{c}$, $r_1 \simeq r_2 \simeq R_e$, so that the Sagnac correction is

$$\delta t \simeq \frac{d_i - d_s}{c} \simeq \frac{\omega_e R_e^2}{c^2}\sin\theta_1\sin\theta_2\sin(\phi_1-\phi_2) \tag{2.19}$$ sagnac

Written in this form, and parametrizing $\vec{r}_2 = \vec{r}_1 + \vec{\Delta r}$ we may rewrite this as

$$\delta t \simeq \frac{\vec{\omega} \cdot (\vec{r}_1 \times \vec{\Delta r})}{c^2} \tag{2.20}$$

sagnac2

We use for Gran Sasso

$$\theta_1 \simeq 47.53^o \quad , \quad \phi_1 \simeq 13.55^o \tag{2.21}$$

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and for CERN

$$\theta_2 \simeq 43.80^o \quad , \quad \phi_2 \simeq 6.15^o \tag{2.22}$$

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Therefore the spherical Sagnac correction is

$$\delta t \simeq 2.16 \; ns \tag{2.23}$$

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This is 6 times smaller than the quoted OPERA error.

3. Schwartzschild geodesics

We will estimate here the effect of the earth gravitational field on the propagation of neutrinos.[1]

The metric produced by the earth is given, to a first approximation, by the Schwarzschild metric:

$$ds^2 = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad , \quad f(r) = 1 - \frac{2GM}{c^2 r} = 1 - \frac{r_s}{r} \tag{3.1}$$

We will consider null geodesics that start at a distance $r = R$ from the center and end up at the same distance after an angle $\Delta\theta$. We will fix $\phi = 0$ by symmetry.

The null geodesics are obtained from the massless particle action

$$S = \int d\tau \left[-f(r)c^2\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\theta}^2 \right] \tag{3.2}$$

where $\dot{r} = \frac{dr}{d\tau}$ and from which the equations for t, θ follow

$$c\dot{t} = \frac{E}{f} \quad , \quad \dot{\theta} = \frac{\mathcal{L}}{r^2} \tag{3.3}$$

where $E, \mathcal{L}$ are two constants of the motion. Instead of deriving the equation for r we integrate the null condition

$$-f(r)c^2\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\theta}^2 = 0, \tag{3.4}$$

null

from which we obtain:

$$\dot{r}^2 = E^2 - \frac{\mathcal{L}^2 f}{r^2} \quad , \quad \frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = cf\sqrt{1 - \frac{\mathcal{L}^2 f}{E^2 r^2}} \quad , \quad \frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{Er^2}{\mathcal{L}}\sqrt{1 - \frac{\mathcal{L}^2 f}{E^2 r^2}}, \tag{3.5}$$

[1]A similar estimate with compatible results was also done in petrolust [3].

Integrating these relations we obtain:

$$c\Delta t = 2\int_{r_{min}}^{R} \frac{dr}{f\sqrt{1-\frac{\mathcal{L}^2 f}{E^2 r^2}}} \quad , \quad \Delta\theta = \frac{2\mathcal{L}}{E}\int_{r_{min}}^{R} \frac{dr}{r^2\sqrt{1-\frac{\mathcal{L}^2 f}{E^2 r^2}}}, \tag{3.6}$$ 22

where Δt is the asymptotic (Schwarzschild) time for the beam to go from source to detector, $\Delta t = t_{out} - t_{in}$ and r_{min} is the minimum distance of the geodesic to the center. The latter is determined by the equation

$$\left.\frac{dr}{d\theta}\right|_{r_{min}} = 0 \quad \Rightarrow \quad \frac{f(r_{min})}{r_{min}^2} = \frac{E^2}{\mathcal{L}^2} \equiv \frac{1}{b^2}. \tag{3.7}$$ 23

The last equation defines the impact parameter b, through which all dependence upon E and $\mathcal{L}$ enters the problem. It is determined by solving simultaneously equation (3.7) together with second equation in (3.6) for a given $\Delta\theta$. For example, for $f(r) = 1$ we get the Euclidean results $b = r_{min} = R\cos(\Delta\theta/2)$.

The length of the geodesic is given by:

$$L_g = \int \sqrt{\frac{dr^2}{f} + r^2 d\theta^2} = \int_{t_{in}}^{t_{out}} dt\sqrt{\frac{1}{f}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2} = \int_{t_{in}}^{t_{out}} cdt\sqrt{f} = 2\int_{r_{min}}^{R} \frac{dr}{\sqrt{f}\sqrt{1-\frac{\mathcal{L}^2 f}{E^2 r^2}}} \tag{3.8}$$

Since $\sqrt{f} \leq 1$, we obtain

$$\frac{L_g}{c\Delta t} \leq 1 \tag{3.9}$$

This is not however the velocity as measured on earth, because an observer on the earth's surface measures a local time interval given by:

$$\Delta t_{\rm local} = \sqrt{f(R)}\Delta t. \tag{3.10}$$

Moreover, the distance used in the measurement was taken to be the distance between the two points along a straight line, and not the distance along the geodesic. The linear (Euclidean) distance between the two points on the surface of the sphere, is

$$L = 2R\sin\frac{\Delta\theta}{2}. \tag{3.11}$$

The length of the same straight line path, but in the Schwarzschild metric, is instead:

$$\frac{dr}{d\theta} = \frac{r^2}{r_{min,o}}\sqrt{1-\frac{r_{min,o}^2}{r^2}} \quad , \quad L_{st} = 2\int_{r_{min,o}}^{R} dr\sqrt{\frac{1}{f} + \frac{1}{r^2/r_{min,o}^2 - 1}} \tag{3.12}$$ 24f

where $r_{min,o} = R\cos\Delta\theta/2$.

In the absence of central mass the motion is rectilinear

$$r(t) = \sqrt{r_{min,o}^2 + t^2} \quad , \quad \tan\theta(t) = \frac{t}{r_{\rm min}}, \tag{3.13}$$ 25f

and the parameters of the motion are given by:

$$b = r_{min,o} = \sqrt{R^2 - L^2/4} = R \cos \Delta\theta/2 \quad , \quad c\Delta t = L = 2R \sin \Delta\theta/2. \tag{3.14}$$

In this case $L = L_g = L_{st}$.

We now consider the first correction due to the non-zero mass, i.e. to the nontrivial potential $f(r)$. All effects are controlled by the quantity

$$\epsilon(r) \equiv f(r) - 1 = \frac{r_s}{r}, \qquad r_s = \frac{2GM}{c^2}, \tag{3.15}$$

Here r_s is the Schwarzschild radius of the earth. Notice that, for the average distance scales involved, $\epsilon(r) \ll 1$, and in fact it is $\approx 10^{-9}$ (for the earth, $r_s \simeq 0.9\ cm$, and $r_{min} \sim R$). Thus we can expand the formulae for distances and time intervals in the small quantity $\epsilon(r)$, and after integration we can express the result as a power series in the small constant parameter[2] r_s/b. In the rest of this section we work to first order in this quantity.

Explicitly, to first order in r_s/b, the observed time of flight is given by:

$$c\Delta t_{local} \simeq 2\sqrt{R^2 - r_{min}^2} + r_s \left\{ \sqrt{\frac{R - r_{min}}{R + r_{min}}} - \frac{\sqrt{R^2 - r_{min}^2}}{R} + 2 \log \frac{R + \sqrt{R^2 - r_{min}^2}}{r_{min}} \right\}. \tag{3.16}$$

To express this result in terms of the euclidean distance L_E, or equivalently $\Delta\theta$, we must determine the correction to r_{min}. This can be achieved by solving perturbatively the second equation in (3.6) and equation (3.7) for b and r_{min} , with $\Delta\theta$ given. One finds:

$$b \simeq r_{min} + \frac{r_s}{2} \quad , \quad r_{min} \simeq R \cos \Delta\theta/2 - \frac{r_s}{2} \left(1 - \frac{1}{\cos \Delta\theta/2} \right). \tag{3.17}$$

This gives an additional correction if we want to compare eq. (3.16) with the zeroth order result $\Delta t = L$. Expanding to linear order the first term in (3.16) around $r_{min} \simeq r_{min,o} \equiv \sqrt{R^2 - L^2/4}$ we arrive at:

$$\delta_e^{(1)} = \frac{L}{\Delta t_{local}} - 1 = -\frac{r_s}{L} \left\{ 3 \frac{R - \sqrt{R^2 - L^2/4}}{L} - \frac{L}{2R} + 2 \log \frac{R + L/2}{\sqrt{R^2 - L^2/4}} \right\} \tag{3.18}$$

If the measured path length is indeed the euclidean one L, then equation (3.18) gives the expected value of a nonzero $(v - c)/c$ along this path due to the earth's gravitational field, to lowest order. On the other hand, one may argue that, as distances are measured locally, the true length of the straight path between CERN and Gran Sasso should be computed using the curved metric, i.e. it should be given by eq. (3.12). To first order, one finds:

$$L_{st} = L - r_s \left\{ \frac{L}{2R} - \cosh^{-1} \frac{R}{\sqrt{R^2 - L^2/4}} \right\} \tag{3.19}$$

[2] One must be careful in not expanding denominators in $\epsilon(r)$ otherwise the integrations will give seemingly non-analytic contributions. Performing the integration in appropriate variables, where the lower extrema are independent of ϵ, one finds an analytic result in r_s/b.

Using this length, the deviation from the speed of light in vacuum has an additional contribution:

$$\delta_e^{(2)} = \frac{L_{st}}{\Delta t_{local}} - 1 = \delta_e^{(1)} - \frac{r_s}{L}\left\{\frac{L}{2R} - \cosh^{-1}\frac{R}{\sqrt{R^2 - L^2/4}}\right\} \qquad (3.20)$$ 28f

As expected, the corrections are controlled by the dimensionless parameter

$$\epsilon = \frac{r_s}{L} = \frac{2GM_e}{Lc^2} \simeq 1.2 \times 10^{-8} \qquad (3.21)$$

where L is the euclidean linear distance of the problem. To obtain this estimate we used the values in (1.1)-(1.3). This effect is therefore three orders of magnitude too small to account for the observed OPERA anomaly that is $\sim 10^{-5}$. For concreteness, the parameters $\delta_e^{(1,2)}$ characterizing the effect of the earth's gravitational field, are given in case of the OPERA beam by:

$$\delta_e^{(1)} \simeq \delta_e^{(2)} = -1.22 \times 10^{-9} \quad , \quad \delta_e^{(2)} - \delta_e^{(1)} \approx \times 10^{-12} \qquad (3.22)$$

where we have set $R = R_e$ and use the numerical values in eqs (2.21-(1.3). Incidentally, we see that the length contraction effect along the straight path is four orders of magnitude smaller than the leading effect.

3.1 Non-inertial effects due to the moon, sun and the galaxy

Using the same method outlined here, we can estimate the effects of the gravitational fields of the moon, sun and of the galaxy. Apart from the earth's rotation around its axis, which was considered in the previous section, the earth is to a very good approximation an inertial reference frame, since it follows a freely falling motion in the external gravitational field generated by all other astronomic objects. Notice that in the case of the moon, sun, and galaxy we have $R \gg L$, and all terms in the brackets in eqs. (3.19) and (3.20) are of order L/R. thus the expansion parameter that controls the effect is now r_s/R, where R is the average distance to the source of the gravitational field To estimate these effect we use the following masses and average distances:

- **moon**

$$M_m \simeq 7.36 \cdot 10^{22}\ Kgr \quad , \quad R_m \simeq 3.84 \cdot 10^{8}\ m \qquad (3.23)$$

- **sun**

$$M_s \simeq 2.00 \cdot 10^{30}\ Kgr \quad , \quad R_s \simeq 1.50 \cdot 10^{11}\ m \qquad (3.24)$$

- **galaxy**

$$M_g \simeq 9 \cdot 10^{10} M_O \simeq 1.8 \cdot 10^{41}\ Kgr \quad , \quad R_g \simeq 8\ kpc \simeq 2.4 \cdot 10^{20}\ m \qquad (3.25)$$

where we used for the galaxy the mass (including the dark matter) inside the sun's orbit around the center, and for R_g the average distance of the solar system to the center of the galaxy.

Using these values we arrive at the following estimates:

$$\delta_m^{(1,2)} \simeq -2.48 \cdot 10^{-13}, \qquad \delta_s^{(1,2)} \simeq -1.73 \cdot 10^{-8}, \qquad \delta_g^{(1,2)} \simeq -4.16 \cdot 10^{-7} \tag{3.26}$$

These are all too small to account for the observed anomaly. Moreover, the sign goes in the opposite direction.

Note that the estimates (3.26) are a (very conservative) upper bound on the effect: since the earth essentially in free fall in the surrounding gravitational fields, the only observable effects will be those due to the *variatios* of the gravitational potential over the distance L travelled by the beam. The real effects thus will be of order $Delta\epsilon$ where

$$\Delta\epsilon \approx \frac{r_s}{R} - \frac{r_s}{R+L} \simeq \left(\frac{r_s}{R}\right)\left(\frac{L}{R}\right). \tag{3.27}$$

The extra factor L/R is tiny: fot the sun, moon and galaxy this adds an extra suppression of 10^{-3}, 10^{-6} and 10^{-15}, respectively.

3.2 Local red-shift correction to the clocks

Two clocks at different places in a gravitational field suffer a red-shift correction obtained from the metric at a single point $ds^2 = g_{tt}c^2dt^2$. This is controlled by $\sqrt{g_{tt}} \simeq 1 - \frac{r_2}{2r} + \cdots$.

Therefore the relative difference of clocks is given by

$$\delta_{12} \equiv \frac{dt_1 - dt_2}{dt_2} = \sqrt{\frac{g_{tt}(1)}{g_{tt}(2)}} - 1 \simeq \frac{r_s(r_1 - r_2)}{2r_1r_2} \tag{3.28}$$

For CERN and Gran-Sasso we can take the conservative estimate $r_1 - r_2 \simeq 1000$ m, and in the denominator $r_1 = r_2 = R_e$, to finally obtain

$$\delta_{12} \simeq 1.1 \cdot 10^{-13} \tag{3.29}$$

4. Earth dipole contributions

The Earth due to rotation is not a perfect sphere. The first correction to the Schwartschild geometry because of this can be written as, [2]

$$ds^2 = -\left(1 + \frac{2V}{c^2}\right)c^2dt^2 + \left(1 - \frac{2V}{c^2}\right)(dr^2 + r^2(d\theta^2 + \sin^2\theta(d\phi + \omega dt)^2)) \tag{4.1}$$

with

$$V = -\frac{GM_e}{r}\left[1 - J_2\frac{R^2}{r^2}P_2(\cos\theta)\right] = V_0 + \delta V \tag{4.2}$$

We have linearized the blackness function as on the surface of the Earth where we are interested, the gravitational potential is very small. $V_0 = -\frac{GM}{r}$ is the standard Schwartzschild answer. We also have

$$J_2 \simeq 1.08 \cdots 10^{-3} \quad , \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \tag{4.3}$$

We may estimate using (2.21) and (2.22)

$$\left|\frac{\delta V}{V_0}\right|_{CERN} \simeq 3 \cdot 10^{-4} \quad , \quad \left|\frac{\delta V}{V_0}\right|_{GS} \simeq 2 \cdot 10^{-4} \tag{4.4}$$

5. Frame-Dragging effects

We will now estimate the order of magnitude of corrections from the frame-dragging effect of general relativity. To do this we must consider the geometry of a rotating body, the Kerr geometry, given by the following metric

$$ds^2 = c^2 d\tau^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\, d\phi^2 + + \frac{2 r_s r \alpha \sin^2\theta}{\rho^2}\, c\, dt\, d\phi \qquad (5.1)$$ 24

with

$$r_s = \frac{2GM_e}{c^2} \quad , \quad \alpha = \frac{J}{M_e c} \quad , \quad \rho^2 = r^2 + \alpha^2\cos^2\theta \quad , \quad \Delta = r^2 - r_s r + \alpha^2, \qquad (5.2)$$ 25

where J is the total angular momentum.

The metric can also be rewritten as

$$ds^2 = \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right) dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi}\left(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}} dt\right)^2. \qquad (5.3)$$ 26

with angular velocity

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{r_s r \alpha c}{\rho^2\left(r^2 + \alpha^2\right) + r_s r \alpha^2 \sin^2\theta}. \qquad (5.4)$$ 27

with which frames drag around the rotating body.

As typically distances are much larger than α, r_s we expand for $r \gg \alpha, r_s$ to obtain

$$-\frac{g_{tt}}{c^2} \simeq 1 - \frac{r_s}{r} + \frac{\alpha^2 r_s}{r^3}\cos^2\theta + \cdots \qquad (5.5)$$ 28

$$g_{rr} \simeq 1 + \frac{r_s}{r} + \frac{r_s^2 - \alpha^2}{r^2} + \frac{\alpha^2}{r^2}\cos^2\theta + \cdots \qquad (5.6)$$ 29

$$g_{\theta\theta} \simeq r^2\left(1 + \frac{\alpha^2}{r^2}\cos^2\theta\right) \quad , \quad \frac{g_{t\phi}}{c} \simeq \frac{2 r_s \alpha}{r}\left(1 - \frac{\alpha^2}{r^2}\sin^2\theta + \cdots\right)\sin^2\theta \qquad (5.7)$$ 30

$$g_{\phi\phi} \simeq r^2 \sin^2\theta\left(1 + \frac{\alpha^2}{r^2} + \frac{r_s \alpha^2}{r^3}\sin^2\theta + \cdots\right) \qquad (5.8)$$ 31

The angular velocity is

$$\Omega = \frac{\alpha r_s c}{r^3} \frac{1}{1 + (1 + \cos^2\theta)\frac{\alpha^2}{r^2} + \frac{r_s \alpha^2}{r^3}\sin^2\theta + \frac{\alpha^4}{r^4}\cos^2\theta} \qquad (5.9)$$ 32

For the earth, $r_s \simeq 9 \cdot 10^{-3}$ m. The approximate angular momentum of a sphere of radius R is

$$J = \frac{2}{5} M_e \omega_e R_e^2 \simeq 7.17 \cdot 10^{33}\ \frac{\text{Kgr m}^2}{\text{sec}} \quad , \quad \alpha = \frac{J}{M_e c} \simeq 3.99\ m \qquad (5.10)$$ 33

Therefore we can estimate on the earth surface the magnitudes

$$, \quad \frac{r_s}{r} \simeq 1.4 \cdot 10^{-9} \quad , \quad \frac{\alpha^2}{R_e^2} \simeq 4 \cdot 10^{-13} \quad , \quad \frac{r_s \alpha^2}{R_e^3} \simeq 5.5 \cdot 10^{-22} \quad , \quad \frac{\alpha^4}{R_e^4} \simeq 1.6 \cdot 10^{-25} \qquad (5.11)$$ 34

We finally obtain

$$\Omega \simeq \frac{\alpha r_s c}{R_e^3} \simeq 4.11 \times 10^{-14} \ \mathrm{sec}^{-1} \qquad (5.12)$$ 35

that gives a negligible effect for our problem.

A. Some useful numbers

We use the following values of relevant parameters

$$\omega_e \simeq 7.3 \times 10^{-5} \quad sec^{-1} \quad , \quad R_e \simeq 6.37 \times 10^6 \quad m \quad , \qquad (1.1)$$ 18

for the earth angular velocity and mean radius,

$$c \simeq 3 \cdot 10^8 \ \frac{m}{sec} \quad , \quad L = 731 \ km \quad , \quad \Delta t_0 = \frac{L}{c} \simeq 2.44 \ ms \qquad (1.2)$$ 118

for the velocity of light in vacuum, the straight distance CERN-Gran Sassso, and the approximate time Δt_0 a light-beam takes to go from CERN to Gran-Sasso. Finally

$$M_e \simeq 5.97 \times 10^{24} \ Kgr \quad , \quad G \simeq 6.67 \times 10^{-11} \ m^3 kgr^{-1} sec^{-2} \quad , \quad r_s = \frac{2GM}{c^2} \simeq 0.9 \ cm \quad (1.3)$$ 119

for the mass of the earth M_e, the gravitational constant G, and the earth Schwartzschild radius r_s.

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